Vacuum polarization on the spinning circle

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(Dated: December 14, 2001)

Vacuum polarization of a massive scalar field in the background of a two-dimensional version of a spinning cosmic string is investigated. It is shown that when the "radius of the universe" is such that spacetime is globally hyperbolic the vacuum fluctuations are well behaved, diverging though on the "chronology horizon". Naive use of the formulas when spacetime is nonglobally hyperbolic leads to unphysical results. It is also pointed out that the set of normal modes used previously in the literature to address the problem gives rise to two-point functions which do not have a Hadamard form, and therefore are not physically acceptable. Such normal modes correspond to a locally (but not globally) Minkowski time, which appears to be at first sight a natural choice of time to implement quantization.

PACS numbers: 04.70.Dy, 04.62.+v

arXiv:gr-qc/0112055 v2 3 Jun 2002

The study of quantum fields around cosmic strings is a pertinent issue since such defects may play a role in the cosmological scenario [1]. Most of the literature concerns spinless cosmic strings (see Ref. [1] and references therein), and only a few works have considered quantum mechanics and quantum field theory around spinning cosmic strings [2, 3, 4, 5, 6, 7].

The locally flat spacetime around an infinitely thin spinning cosmic string [1] is characterized by the line element

$$ds^2 = (d\tau + Sd\varphi)^2 - d\rho^2 - \alpha^2\rho^2 d\varphi^2 - dz^2 \qquad (1)$$

and by the identification $(\tau, \rho, \varphi, z) \sim (\tau, \rho, \varphi + 2\pi, z)$, where $0 < \alpha \leq 1$ is the cone parameter and $S \geq 0$ is the spin density (clearly the Minkowski spacetime corresponds to $S = 0$ and $\alpha = 1$). As the region for which $\rho < S/\alpha$ contains closed timelike contours the spacetime is not globally hyperbolic. In other words a global time is not available, so that it is not clear whether quantum theory makes sense in this background [8].

The study of a relativistic quantum scalar particle moving on the spinning cone [the corresponding three-dimensional line element is obtained from Eq. (1) by setting $dz = 0$] has shown that a nonvanishing S spoils unitarity [2]. It has been speculated that this sort of first quantized pathology could be eliminated in the second quantized approach. However, Ref. [7] seems to frustrate this possibility by showing that the vacuum fluctuations of a massless scalar field diverge on concentric cylindrical shells around the spinning cosmic string. These pathological results have been attributed to the nonglobally hyperbolic nature of the background.

In order to exhibit clearly aspects of global hyperbolicity (and related issues) in actual calculations, this work will consider a toy model which consists of a quantum scalar field existing in a two dimensional spacetime whose line element is obtained by truncating Eq. (1) as [5]

$$ds^2 = (d\tau + Sd\varphi)^2 - \rho^2 d\varphi^2 \qquad (2)$$

(α was dropped since it can be removed by redefining the parameter ρ) and observing

$$(\tau, \varphi) \sim (\tau, \varphi + 2\pi)\,. \qquad (3)$$

It is clear that $S = 0$ corresponds to a cylindrical spacetime of periodicity length $2\pi\rho$. The main pedagogical feature of this toy model is that, for a given "radius of the universe" $\rho > 0$, one can tune the spin S such that the background is globally hyperbolic ($\rho > S$) or otherwise ($\rho \leq S$).

Assuming $\rho > S$ and defining new parameters

$$r := \sqrt{\rho^2 - S^2}, \qquad \Omega := \frac{S}{\rho\sqrt{\rho^2 - S^2}} \qquad (4)$$

and a new time coordinate

$$t := \frac{\tau}{\sqrt{1 - \Omega^2 r^2}}, \qquad (5)$$

it follows that Eq. (2) can be recast as

$$ds^2 = (1 - \Omega^2 r^2)dt^2 + 2\Omega r^2 dt\, d\varphi - r^2 d\varphi^2. \qquad (6)$$

Defining further $\theta := \varphi - \Omega t$ Eq. (6) becomes

$$ds^2 = dt^2 - dx^2, \qquad (7)$$

with $x := r\theta$, and Eq. (3) leads to

$$(t, x) \sim (t,\ x + L), \qquad (8)$$

where

$$L := 2\pi\sqrt{\rho^2 - S^2}. \qquad (9)$$

Therefore the spinning circle defined in Eq. (2) is a cylindrical spacetime with the spin S encapsulated in the periodicity length L.

Alternatively, Eq. (6) describes the background as a cylindrical spacetime seen from a rotating frame with angular velocity $-\Omega$. Noting that $\rho > S$ leads to $r > 0$ and consequently $\Omega r < 1$, by requiring global hyperbolicity one ensures that the velocity of the rotating frame is less than the velocity of light. It should also be remarked that as $S \to \rho$, $r \to 0$ and $\Omega r \to 1$. That is, on the "chronology horizon" ($\rho = S$) the corresponding cylindrical spacetime collapses to its axis and the associated rotating frame reaches the velocity of light.

As long as $\rho > S$, i.e. the spacetime is globally hyperbolic, one can implement quantization in any of the frames considered above [corresponding to Eqs. (2), (6), and (7)], since the time coordinates are genuine global times (in the sense that they parametrize Cauchy surfaces) and the corresponding time translation Killing vectors are globally timelike. The usual procedure to quantize a scalar field $\phi(\mathrm{x})$ reveals that these frames have identical sets of normal modes, and therefore identical vacuum states (which is not surprising since there is no event horizon involved [9]). It is clear from Eqs. (7) and (8) that the set of normal modes is that associated with a cylindrical two-dimensional spacetime, which is well known in the literature [10, 11].

The Hadamard function corresponding to the field modes appearing in Eq. (4.1) of Ref. [10] [with $\omega = (k^2+m^2)^{1/2}$] can be cast as

$$G^{(1)}(\mathrm{x},\mathrm{x}') = \frac{1}{mL} - \frac{1}{4\pi}\ln\left\{4\left[\cos(2\pi\Delta t/L) - \cos(2\pi\Delta x/L)\right]^2\right\}, \tag{10}$$

where $\Delta t := t - t'$, $\Delta x := x - x'$, and a small mass m was taken into account to prevent the usual infrared divergence [in fact $mL << 1$ has been considered and, accordingly, higher powers of mL were omitted in Eq. (10)]. Incidentally one may check that the massless contribution in Eq. (10) reproduces Eq. (4.23) in Ref. [10].

A quick examination of Eqs. (9) and (10) shows that $G^{(1)}(\mathrm{x},\mathrm{x}')$ diverges when $\rho = S$. As the vacuum fluctuations $\langle\phi^2(\mathrm{x})\rangle$ can be formally obtained from the Hadamard function [10, 12], i.e.,

$$\langle\phi^2(\mathrm{x})\rangle = \lim_{\mathrm{x}'\to\mathrm{x}} \frac{1}{2} G^{(1)}(\mathrm{x},\mathrm{x}'), \tag{11}$$

one may wonder whether $\langle\phi^2(\mathrm{x})\rangle$ itself diverges on the "chronology horizon". That is indeed the case as the following calculations show.

As the background is flat, the ultraviolet divergence arising in Eq. (11) can be cured simply by removing the contribution in Minkowski spacetime. It follows from Eq. (10) that its short distance behavior is given by

$$G^{(1)}(\mathrm{x},\mathrm{x}') = \frac{1}{mL} - \frac{1}{2\pi}\ln(4\pi^2|\sigma|/L^2), \tag{12}$$

where $\sigma := (\Delta t)^2 - (\Delta x)^2$. The Minkowski contribution is given by [10]

$$G_0^{(1)}(\mathrm{x},\mathrm{x}') = -\frac{1}{2\pi}\left[\ln(m^2|\sigma|/4) + 2\gamma\right], \tag{13}$$

with γ denoting the Euler constant. By subtracting Eq. (13) from Eq. (12) before taking the limit in Eq. (11), one finds that

$$\langle\phi^2(\mathrm{x})\rangle = \frac{1}{2mL} + \frac{1}{2\pi}\left[\ln(mL/4\pi) + \gamma\right], \tag{14}$$

where the usual infrared divergence ($m = 0$) and the chronology divergence ($L = 0$) have identical structures. It should also be pointed out that the vacuum fluctuations lose reality when Eq. (14) is naively used when $\rho < S$ [cf. Eq. (9)].

The procedure outlined above can be extended to evaluate vacuum expectation values of other quantities such as the components of the energy momentum tensor. The energy density, the pressure, and the momentum density are given, respectively, by $-\pi/6L^2 + m/2L$, $-\pi/6L^2$, and 0, with respect to the frame corresponding to Eq. (7) [10, 11]. By evoking Eq. (9) one sees that the energy momentum tensor also diverges on the "chronology horizon". (It is worth remarking that the spin S does not induce momentum density, which is reasonable since $S \neq 0$ affects L only.)

Considerations on a certain "time-helical" structure [13] are in order. By redefining the time coordinate according to

$$T := \tau + S\varphi \tag{15}$$

Eq. (2) can be recast in Minkowski form,

$$ds^2 = dT^2 - dX^2, \tag{16}$$

where $X := \rho\varphi$. Observing Eqs. (3) and (15), it follows that

$$(T, X) \sim (T + 2\pi S,\ X + \mathrm{L}), \tag{17}$$

with $\mathrm{L} := 2\pi\rho$. Equations (16) and (17) should be compared with Eqs. (7) and (8), respectively. The main difference is that T satisfies an unusual identification, giving rise to a "time-helical" structure.

The time coordinates T and t are related by [cf. Eq. (5)]

$$T = \frac{t - Vx}{\sqrt{1 - V^2}}, \tag{18}$$

where $V := -\Omega r$. If x were to label points on the line Eq. (18) would be identified as a genuine Lorentz transformation. Nevertheless, x labels points on the circle [cf.

Eq. (8)] and Eq. (18) can only be considered as a Lorentz transformation locally, i.e., when T, t, and x are replaced by dT, dt, and dx, respectively. It turns out that (when $V \neq 0$) the "helical time" T is locally Minkowski only, whereas t is globally Minkowski.

The time coordinate T has been used in Ref. [5] to quantize a scalar field on the spinning circle. Although the background is flat, the corresponding two-point function presents (in addition to the usual flat divergence) short distance divergences containing the spin S as a factor. Such divergences certainly cannot be renormalized away by subtracting the Minkowski contribution, and one says that the two-point function does not have the Hadamard form [12]. This unphysical feature is not surprising if one recalls that the standard knowledge requires a global time in implementing quantization. In fact, T is not a global time, since constant values of T do not parametrize Cauchy surfaces as long as $S \neq 0$ [cf. Eq. (17)]. Therefore the results in Ref. [5] are spoiled by improper use of T as a global time. [It should be mentioned that improper use of "helical times" as global times may also spoil results in other contexts. For instance, the use of T as given by Eq. (18) to study the propagation of light in a rotating frame yields results that contradict well established experimental facts [14].]

Summarizing, this toy model illustrated in actual calculations the relevance of global hyperbolicity for a consistent quantization, and some consequences of the improper use of "helical times" to address global issues.

Acknowledgments

The authors are grateful to George Matsas, Raphael De Paola, Renato Klippert, and Ricardo Medina for useful discussions. This work was partially supported by the Brazilian research council CNPq.

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